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On weighted sums in abelian groups

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Abstract

Let G be an abelian group of order n and Davenport constant d and let k be a natural number. Let $x_0, x_1, \dots, x_m$ be a sequence of elements of G such that x_0 has the most repeated value in the sequence. Let $\{w_i;\ 1 \leqslant i \leqslant k\}$ be a family of integers prime relative to n. We obtain the following two generalizations of the Erdős–Ginzburg–Ziv Theorem.

For $m \geqslant n + k - 1$, we prove that there is a permutation α of $[1, m]$ such that

$$\sum_{1 \leqslant i \leqslant k} w_i\, x_{\alpha(i)} = \left(\sum_{1 \leqslant i \leqslant k} w_i \right) x_0.$$

For $k \geqslant n - 1$ and $m \geqslant k + d - 1$, we prove that there is a k-subset $K \subset [1, m]$ such that

$$\sum_{i \in K} x_i = k x_0.$$

1. Introduction

The Davenport constant $D(G)$ of an abelian group G is the minimal k such that every sequence of elements of G with length k contains a nonempty subsequence with a zero sum. The Erdős–Ginzburg–Ziv Theorem [4] states that every sequence in an abelian group of order n having length $2n - 1$ contains a subsequence with length n and zero sum. Some of the existing proofs of this result are contained in [1].

Olson generalized this result in [9], by showing that under the hypothesis $n \geqslant 2$ there is a nonnull subgroup each element of which is a sum of a subsequence of length n, unless there is a value repeated $(n + 1)$ times. Notice that Olson's Theorem applies also to nonabelian groups if one allows reordering of the sequence elements, cf. [9].

A recent result of Weidong unifies the above two topics of zero sum problems. This result is independently conjectured and proved for p-groups by Caro in [3]. Weidong's Theorem states that every sequence of length $|G| + D(G) - 1$ contains a subsequence with length $|G|$ and zero sum, cf. [10]. In particular the Erdős–Ginzburg–Ziv Theorem follows using the obvious bound $D(G) \leqslant |G|$.

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Weighted generalizations of the Erdős–Ginzburg–Ziv Theorem were considered recently. The first one is a conjecture of Caro, proved by Alon in prime order, cf. [2]. In [7], we could proof a weighted generalization of Olson's Theorem mentioned above.

We prove in Section 2 the following result.

Let G be an abelian group of order n and let k be a natural number. Let $x_0, x_1, \ldots, x_{n+k-1}$ be a sequence of elements of G such that x_0 has the most repeated value in the sequence. Let $\{w_i;\ 1 \leqslant i \leqslant k\}$ be a family of integers prime relative to n.

Then there is a permutation α of $[1, n+k-1]$ such that

$$\sum_{1 \leqslant i \leqslant k} w_i\, x_{\alpha(i)} = \left(\sum_{1 \leqslant i \leqslant k} w_i \right) x_0.$$

We could not conjecture the validity of this result if some weights are not prime relative to n.

In Section 3, we consider the case where all the weights are equal. We generalize the result of Weidong mentioned above as follows.

Let G be an abelian group with a Davenport constant d and let $k \geqslant |G| - 1$ be a natural number. Let $x_1, x_2, \ldots, x_{d+k}$ be a sequence of elements of G such that x_1 has the most repeated value in the sequence.

Then there is a k- subset $K \subset [2, d+k]$ such that

$$\sum_{i \in K} x_i = k x_1.$$

We use the following easy consequence of Proposition 2.2 of [6].

Lemma 1.1. *Let G be a finite abelian group and let $\{a_i;\ 1 \leqslant i \leqslant |G|\}$ be a sequence of elements of of G in which no value is repeated $(s+1)$-times. Let $\{w_i;\ 1 \leqslant i \leqslant s\}$ be a family of integers prime relative to n and let $x \in G$. There are a non empty $K \subset [1, s]$ and a permutation τ of $[1, n]$ such that*

$$\sum_{i \in K} w_i a_{\tau(i)} = \sum_{i \in K} w_i x.$$

Proof. We may clearly assume $s \leqslant n$. Take a partition of $\{a_i;\ 1 \leqslant i \leqslant |G|\}$ into s nonempty sets $\{A_i;\ 1 \leqslant i \leqslant s\}$. We have clearly

$$|w_1(A_1 - x)| + , \ldots , + |w_s(A_s - x)| = |A_1| + , \ldots , + |A_s| = n.$$

By Proposition 2.2 of [6], there is a non empty $K \subset [1, s]$ such that $0 \in \sum_{i \in K} w_i(A_i - x)$. $\square$

2. Weighted sums

The main result in this section is the following one.

Theorem 2.1. *Let G be an abelian group of order n and let k be a natural number. Let $x_0, x_1, \ldots, x_{n+k-1}$ be a sequence of elements of G such that x_0 has the most repeated value in the sequence. Let $\{w_i;\ 1 \leqslant i \leqslant k\}$ be a family of integers prime relative to n.*

Then there is a permutation α of $[1, n+k-1]$ such that

$$\sum_{1 \leqslant i \leqslant k} w_i\, x_{\alpha(i)} = \left(\sum_{1 \leqslant i \leqslant k} w_i\right) x_0.$$

Proof. We represent the sequence by a mapping $f : [0, n+k-1] \to G$, where $f(i) = x_i$, $0 \leqslant i \leqslant n+k-1$.

Set $M = \{i\colon x_i = x_0\}$ and $s = |M|$. The result is obvious if $k \leqslant s+1$. Suppose that $k \geqslant s$.

Consider a maximal subset T contained in $[1, k]$ such that there is an injection $\mu : T \to [1, n+k-1] \setminus M$ satisfying the condition:

$$\sum_{i \in T} w_i f(i) = \sum_{i \in T} w_i x_0.$$

Let us prove that

$$|T| \geqslant k - s + 1 \tag{2}$$

Assume the contrary and put $B = [1, n+k-1] \setminus (\mu(T) \cup M)$. We have clearly $|B| \geqslant n + k - 1 - (s-1) - (k-s) = n$. We have also $|[1,k] \setminus T| \leqslant s$. By Lemma 1.1, applied with $x = x_0$ to the sequence $\{x_i;\ i \in B\}$ and the weights $\{w_i;\ i \in [1,k] \setminus T\}$, there is a nonempty $K \subset [1,k] \setminus T$ and distinct elements $\{b_i;\ i \in K\}$ in B such that

$$\sum_{i \in K} w_i f(b_i) = \sum_{i \in K} w_i x_0. \tag{1}$$

We may check easily that μ extends to an injection $\mu : T \cup K \to [1, n+k-1]$, by putting $\mu(i) = b_i$, for every $i \in K$, contradicting the maximality of T.

Set $W = [1,k] \setminus T$. We have clearly $0 \leqslant |W| \leqslant s - 1 = |M \setminus \{0\}|$. It follows that μ extends to an injection ν from $[1,k]$ into $[1, n+k-1]$ such that

$$\forall i \in W, \nu(i) \in M \setminus \{0\}. \tag{3}$$

Clearly, we have, using (1) and (3),

$$\sum_{i \in [1,k]} w_i\ f(\nu(i)) = \sum_{i \in T} w_i x_0 + \sum_{i \in W} w_i x_0 = \sum_{i \in [1,k]} w_i x_0.$$

This proves the theorem with any permutation α of $[1, n+k-1]$ verifying relation $\alpha(i) = \nu(i)$, $1 \leqslant i \leqslant k$. $\square$

In the case of weights prime relative to n, the conjecture of Caro mentioned in the introduction, follows from Theorem 2.1.

3. Constant weights

In this section, we improve Theorem 2.1, in the case of equal weights and non cyclic groups. Without loss of generality, we may assume all the weights $= 1$.

Lemma 3.1. *Let m, t, m_1 be natural numbers such that $m \geqslant \max(m_1, t)$. Let $m_2, \dots, m_j$ be natural numbers not exceeding $m_1 + 1$ such that*

$$m \geqslant m_1 m \leqslant t + m_1 + m_2 + \cdots + m_j.$$

Then there is $J \subset \{2, \dots, j\}$ and $y \leqslant m_1$ such that

$$m = t + y + \sum_{i \in J} m_i.$$

The proof is easy.

Theorem 3.2. *Let G be an abelian group with a Davenport constant d and let $k \geqslant |G| - 1$ be a natural number. Let $x_1, x_2, \dots, x_{d+k}$ be a sequence of elements of G such that x_1 has the most repeated value in the sequence. Then there is a k-subset $K \subset [2, d+k]$ such that*

$$\sum_{i \in K} x_i = k x_1.$$

Proof. Set $|G| = n$. We represent the sequence by a mapping $f: [1, d+k] \to G$, where $f(i) = x_i$, $1 \leqslant i \leqslant d + k$. Set $M = \{i: x_i = x_1\}$ and $s = |M|$. The result is obvious if $k + 1 \leqslant s$. Suppose that $s \leqslant k$. The result holds trivially if $n = 1$. Assume $n \geqslant 2$. We have clearly $s \geqslant 2$.

Consider a subset $S \subset [1, k+d]$ with maximal cardinality verifying the following conditions (satisfied clearly with $S = M \backslash 1$).

(i) There is a partition $S = \bigcup_{0 \leqslant i \leqslant m} S_i$, where $S_0 = M \backslash 1$, $0 \leqslant m$ and for all i, $1 \leqslant |S_i| \leqslant s$.

(ii) $\sum_{x \in S_i} f(x) = |S_i| x_1$, for all i.

Consider first the case $|S| \geqslant k$. We have clearly $|M \backslash 1| + \sum_{1 \leqslant i \leqslant m} |S_i| \geqslant k$.

We have also $|S_i| \leqslant s \leqslant k$.

By Lemma 3.1 there is $X \subset M \backslash 1$ and $L \subset [1, m]$ such that

$$|X| + \sum_{i \in L} |S_i| = k.$$

Put $A = X \cup \bigcup_{i \in L} S_i$. We have

$$\sum_{x \in A} f(x) = \sum_{x \in X} f(x) + \sum_{i \in L} |S_i| x_1 = k x_1.$$

Assume now that

$$|S| \leqslant k - 1. \tag{1}$$

Let us prove that

$$|S| \geqslant d. \tag{2}$$

Assume the contrary and set $B = [1, d + k - 1] \setminus S$. We have clearly $|B| \geqslant d + k - 1 - (d - 1) = k \geqslant n$. By Lemma 1.1, there is $C \subset B$ such that $1 \leqslant |C| \leqslant s$ and $\sum_{x \in C} f(x) = |C| x_1$.

The partition $C \cup T = C \cup \bigcup_{0 \leqslant i \leqslant m} S_i$ verifies conditions (i) and (ii), contradicting the maximality of S. This proves (2).

Choose a maximal family $\{T_i;\ 1 \leqslant i \leqslant q\}$ of subsets of $[2, d + k] \setminus S$ such that $\sum_{x \in T_i} f(x) = |T_i| x_1$ and $1 \leqslant |T_i| \leqslant d$, for all i.
By the definition of d, we have $|S| + \sum_{1 \leqslant i \leqslant q} |T_i| \geqslant k$.

Choose a minimal $J \subset [1,q]$ such that $|S| + \sum_{i \in J} |T_i| \geqslant k$. Set $T = \bigcup_{i \in J} T_i$. Choose any $j \in J$, notice that J is nonempty by (2). We have $|S| + |T| - |T_j| \leqslant k - 1$. By (2), $|T| \leqslant k - 1 - |S| + |T_j| \leqslant k - |S| + d - 1 \leqslant k - 1$.

Recall that $|S_i| \leqslant s \leqslant k$, for all i and that $|S_0| = s - 1$. Now we have $|T| + |M \setminus 1| + \sum_{1 \leqslant i \leqslant m} |S_i| \geqslant k$.

By Lemma 3.1 there is $X \subset M \setminus 1$ and $L \subset [1, m]$ such that $|T| + |M| + \sum_{1 \leqslant i \leqslant m} |S_i| \geqslant k$.

Put $A = T \cup X \cup \bigcup_{i \in L} S_i$.

We have

$$\sum_{x \in A} f(x) = \sum_{x \in T} f(x) + \sum_{x \in X} f(x) + \sum_{i \in J,\, x \in S_i} f(x).$$

Hence,

$$\sum_{x \in A} f(x) = |T| x_1 + |X| x_1 + \sum_{i \in L} |S_i| x_1 = k x_1.$$

This proves the theorem. □

Corollary 3.3 (Weidong [10]). *Let G be an abelian group with order n and Davenport constant d. Let $x_1, x_2, \ldots, x_{n+d-1}$ be a sequence of elements of G.*
Then there is $K \subset [1, n + d - 1]$ such that $|K| = n$ and

$$\sum_{i \in K} x_i = 0.$$

Proof. We may assume, without loss of generality, that x_1 is the most repeated value in the sequence. By Theorem 3.1, there is $L \subset [2, n + d - 1]$ such that $|L| = n - 1$ and

$$\sum_{i \in L} x_i = -x_1.$$

The result is now obvious. □

References

[1] N. Alon and M. Dubiner, Zero-sum sets of prescribed size, In Combinatorics: Paul Erdős is Eighty, Coll Math. János Bolyai (1993) 33–50.
[2] Y. Caro, Zero-sum Ramsey problems, a survey, Discrete Math., to appear.
[3] Y. Caro, On zero-sum subsequences in abelian non-cyclic groups, Israel Jour. of Mathematics, to appear.
[4] P. Erdős, A. Ginzburg and A. Ziv, A theorem in additive number Theory, Bull. Res. Council, Israel, 10 (Aug. 1961).
[5] Z. Füredi and D.J. Kleitmann, The minimal number of zero sums, Combinatorics, P. Erdős is Eighty Vol. 1, Keszthely (1993) 159–172.
[6] Y.O. Hamidoune, On a subgroup contained in some words with a bounded length, Discrete Math. 103 (1992) 171–176.
[7] Y.O. Hamidoune, On weighted sequence sums, Combin. Algorithms Probab. to appear.
[8] H.B. Mann, Addition Theorems, (Wiley, New York, 1965).
[9] J.E. Olson, A combinatorial problem of Erdős–Ginzburg-Ziv, J.Number Theory 8 (1976) 52–57.
[10] Gao Weidong, Some problems in Additive group theory and additive number theory, Ph.D. Dissertation (Abstract), 1994.
[11] Gao Weidong, An addition theorem for finite cyclic groups, preprint, 1994.